Classical Analysis Techniques Set the Stage for Mastery of Computer Analysis Techniques

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Abstract

This paper describes the successful use of classical analysis techniques by the ABET-accredited CE program at the U.S. Military Academy to assist students in unlocking the mysteries embedded in commercial structural analysis programs that are based on the Direct Stiffness Method. We believe that students must understand the basic assumptions inherent in the classical methods that subtly choreograph structural behavior before they can confidently and competently perform black box structural analyses. We find that students understand these assumptions best when they have an opportunity to work through each of the classical methods prior to attacking the Direct Stiffness Method by hand—aided by appropriate software to perform computations and matrix manipulations, and then by comparing the classical analysis results to commercially available structural analysis programs.

In our first Structural Analysis course at the U.S. Military Academy, students explore structural analysis through the use of both classical (Virtual Work, Slope Deflection, and Moment Distribution) and commercially available structural analysis programs. Each classical method highlights key basics of structural behavior and sets the stage for developing connectivity between the classical hand methods and today’s computer techniques. In our Advanced Structural Analysis course, students learn and apply the Direct Stiffness Method in three different blocks of instruction—Trusses, Beams, and Frames. In each block, we develop the direct stiffness formulation of the appropriate structural element, many times using a classical analysis technique to complete the structural element development, and then we have the students work through one or more problems involving the analysis of a relatively simple structure. In every case, the students analyze the same structure using classical and computer-based applications. The classical methods set the stage for walking through the Direct Stiffness Method inherent in most commercial analysis packages through the use of Excel spreadsheet software to perform matrix manipulations and MathCAD computational software to perform mathematical computations required as part of the Direct Stiffness Method. The results from the classical methods, manual direct stiffness analysis and commercial applications are compared to more fully provide connectivity between the techniques and inherent assumptions.

This paper describes the courses and the use of the classical methods to provide insight into the computer-based black box analysis packages. It also provides an example of a typical student homework problem as well as student assessment data demonstrating the
effectiveness of the methodology in promoting better understanding of: (1) the Direct Stiffness Method itself; (2) the relationship between the Direct Stiffness Method and classical structural analysis techniques like Slope Deflection and Moment Distribution.

I. Introduction

Although a computer will undoubtedly provide results much faster than performing one of the classical structural analysis methods by hand, students need to have an intuition of how structures behave under loading. This intuition is critical to providing a common sense check to both classical and computer generated results and is what really separates an engineer from someone who is simply inputting data into a computer program. Proper understanding of structural behavior is rooted in the basics of support conditions, connections and member properties.

We start the understanding of structural behavior with general drawings of deflected shapes. By analyzing the deflected shape as a curve with boundary conditions, we then discuss resistance to rotation and displacement due to support restraints and member properties. Relating the moment diagram to concavity and inflection points in the deflected shape provides insight into the element’s deflected shape between connections and supports. It is then important to tie in the member properties and structural stiffness as students must learn which member properties affect the structural resistance to bending or axial deformation.

All of these factors are relevant and essential to the professional development of an engineer and are best learned by studying several of the classical methods of structural analysis. In a student’s first structural analysis course at the U.S. Military Academy, indeterminate beams, trusses, and frames are analyzed using virtual work, slope deflection, and moment distribution. These methods are coupled with the use of commercially available structural analysis programs to reinforce the students’ knowledge of key aspects of structural behavior prior to studying the Direct Stiffness Method which is inherent in the commercially available structural analysis programs.

II. Classical Structural Analysis

II.A. Virtual Work

Students go through the derivation of the virtual work equations¹ (EQN 1,2) to gain an understanding of how member properties such as length, cross-sectional area, modulus of elasticity and moment of inertia affect external deflection of a point on a structure.

\[
\text{Deflections in a truss} \quad 1K \cdot \Delta = \sum \eta \frac{NL}{AE} \quad (1)
\]

\[
\text{Deflections in a beam} \quad 1K \cdot \Delta = \int \frac{mM}{EI} dx \quad (2)
\]
The students solve for deflections of trusses, beams, and frames using virtual work. The practical application of these equations allows the students to experiment individually with each property to see how they affect the structural deflection. Deflection results are compared with those obtained using structural analysis programs. Students learn how to manipulate the computer input to provide output at the desired locations, while also experiencing the importance of proper input into the black box analysis program when results do not match.

II.B. Slope Deflection

The classical slope deflection equations are derived based on the students’ knowledge of member loading, material properties, and degrees of freedom (DOF) at end nodes of each member. The following diagram (Figure 1) is used to derive the slope deflection equations (EQN 3,4).

![Figure 1. Externally Loaded Beam and Possible Resulting Shape](image)

\[
M_{zi} = \frac{2EI}{L} \left[ 2\theta_i + \theta_j - 3 \left( \frac{v_j - v_i}{L} \right) \right] + FEM_{ji} \\
M_{zj} = \frac{2EI}{L} \left[ 2\theta_j + \theta_i - 3 \left( \frac{v_j - v_i}{L} \right) \right] + FEM_{ji}
\]
Students learn how to systematically apply these equations to successive members of a structure, resulting in a series of equations that almost mathematically model the entire structure. As is usually the case, there are more unknowns (DOF & internal moments) than equations that require the development of compatibility equations based on end support conditions and internal connectivity of the structure over supports as well as defining known DOFs, such as rotation at fixed boundary conditions and deflections at rollers. These equations are then solved simultaneously using a hand-held calculator or a mathematical solver (e.g., MathCAD). The solution yields the internal moments and applicable nodal rotations. Like the virtual work method, the slope deflection method furthers the students’ understanding of how member properties, as well as connection and support conditions affect a structure. Fixity and larger area moment of inertia decrease rotation and result in larger available moment capacity. Students mathematically see how stiffness attracts load and decreases both rotation and ultimately deflection. The slope deflection equations also provide the theoretical basis for the moment distribution process.

II.C. Moment Distribution

The moment distribution process is the last classical method the students learn prior to the introduction of the direct stiffness method. During the first structural analysis class, the “propped cantilever” beam (Figure 2) serves as the basis for moment distribution. The stiffness and carry-over factors are both taken directly from the slope deflection equations associated with this member.

![Figure 2. Propped Cantilever with an External Moment at the Roller Support](image)

\[
M_{AB} = \frac{4EI\theta_A}{L} \\
M_{BA} = \frac{2EI\theta_A}{L}
\]

The stiffness factor, \( K = \frac{4EI}{L} \) (for the prismatic beam, Figure 2), is the amount of moment required to rotate end A of the beam one radian. Distribution factors are calculated based on a member’s fraction of the overall stiffness at a node. Students see once again how the material properties affect the member stiffness and how member stiffness affects overall structural behavior. The carry-over factor, used to identify how a moment is “transferred” from one end of a beam to the other, also comes from the slope deflections above. The carry-over factor of 0.5 is simply the ratio of \( M_{BA} \) and \( M_{AB} \). Class problems, along with physical models (Figures 3 and 4), are used to demonstrate how the moment distribution process works. Students are exposed to more complex situations in which non-prismatic beams are analyzed in our advanced structural analysis course.
The “moment distributor” begins with all nodes in the locked (fixed) position. From the calculated fixed end moments, it is possible to see that the end moments from each member at a node do not equal zero.

An individual node is released, allowed to reach equilibrium, and is then relocked. This iterative process of distributing moment continues until the distributions are relatively small. This model allows students see how stiffer members attract moments and how moments are distributed as they “travel” through a structure. The same behavior is observed while mathematically completing a moment distribution table using Excel, and observing the column moment totals on each side of a node as each node is iteratively unlocked and relocked sequentially across the structure.

The concepts learned from these classical methods provide the necessary connectivity and knowledge base for transition to the direct stiffness method. By this point, students have a fairly broad knowledge base of how member properties, degrees of freedom, and loading all affect internal member shear forces and moments. These concepts naturally come together during introduction of the direct stiffness method (EQN 5).

III. Homework Problem

After review of the classical stiffness methods discussed above, a student is expected to use the appropriate method based on the situation. Given the problem below (Figure 5), the student is expected to observe how to account for degrees of freedom, loading and settlement in each method, the limitations of each method, and also the common threads concerning degrees of freedom, loading and element stiffness.
Take the example:

The beam below is prismatic between the nodes & continuous over the nodes (E = 29000 ksi, I_{AB}=1000 \text{ in}^4; I_{BC}=I_{CD}=2500 \text{ in}^4). It is loaded as shown. **Node C settles 0.25 inches.**

![Beam Diagram](image)

**Figure 3. In-Class Example Problem**

**III.A. Slope Deflection**

Since slope deflection only solves directly for the internal moments at the ends of each member, we need 2 equations for each member. The resulting six moment equilibrium equations represent M_{AB}, M_{BA}, M_{BC}, M_{CB}, M_{CD} and M_{DC}.

We can quickly set up the two equations for member AB using EQNs 3 and 4 to represent M_{AB} (EQN 6) and M_{BA} (EQN 7).

\[
M_{AB} = \frac{2EI_{AB}}{L_{AB}} \left[ 2\theta_A + \theta_B - 3 \left( \frac{v_B - v_A}{L_{AB}} \right) \right] + \text{FEM}_{AB} \tag{6}
\]

\[
M_{BA} = \frac{2EI_{AB}}{L_{AB}} \left[ 2\theta_B + \theta_A - 3 \left( \frac{v_B - v_A}{L_{AB}} \right) \right] + \text{FEM}_{BA} \tag{7}
\]

Following the same format for the next 4 equations, these 20 apparent unknowns would be developed:

- Internal Moments: M_{AB}, M_{BA}, M_{BC}, M_{CB}, M_{CD} and M_{DC}
- Degrees of Freedom: \theta_A, \theta_B, \theta_C, and \theta_D,
- \(v_A, v_B, v_C, \text{ and } v_D\)
- Fixed End Effects: \text{FEM}_{AB}, \text{FEM}_{BA}, \text{FEM}_{BC}, \text{FEM}_{CB}, \text{FEM}_{CD} \text{ and } \text{FEM}_{DC}
However, some of these symbolically represented unknowns are really known and to solve the problem we must accurately reflect which values are solved for directly using some basics about connections and support conditions.

Some basic engineering tables are used to determine the values for the fixed end effects based on the loading on each segment (AB, BC, and CD). The values for the six fixed end effects (FEM) reduce the total unknowns to 14. At this time, the students must begin to analyze the structure for what they do know.

Depending on the support conditions, the degrees of freedom, DOF, are either known or unknown. In a two dimensional system, there is possible rotation, $\theta$, at the node and translation in both the x and y. For horizontal beam problems with no horizontal loads, the axial element is disregarded (insignificant effects) in our studies at this time. Therefore the student must determine if they know either the rotation or the translation, $v$, in the y-direction. A quick look at the problem statement and the supports as modeled in the problem (Figure 5) help determine several additional known values. Horizontal rollers restrict vertical displacement except in cases of known settlement. Settlement at a roller forces displacement in the system at the particular roller. A fixed support prevents any translation or rotation. Armed with this knowledge, it is possible to declare $\theta_D$, $v_A$, $v_B$, & $v_D$ equal to zero. Due to settlement, $v_C$ = -0.25 inches.

There are now only nine unknowns: $M_{AB}$, $M_{BA}$, $M_{BC}$, $M_{CB}$, $M_{CD}$, $M_{DC}$, $\theta_A$, $\theta_B$ & $\theta_C$, but we still only have six equations. We’ve reduced the unknowns by analyzing the DOF and the loading conditions. Taking a closer look at the internal moments reveals some additional information. The structure is continuous from A to D. Therefore cutting the member at any point would expose equal and opposite moments on each face of the cut if no external applied moment is present at that point. We can cut anywhere along the continuous beam. If we choose to cut the member at B, we can say $M_{BA} = -M_{BC}$. We either add another equation (which we choose to do), or go from two unknowns to one in this case. It is possible to make the same case for equilibrium at C. Hence, $M_{CB} = -M_{CD}$.

At this time, we are left with 8 equations and 9 unknowns. Application of basic statics and the definition of an end roller allows us to also declare $M_{AB} = 0$. A common error amongst students is to quickly get rid of the entire equation for the moment at A.

$$M_{AB} = \frac{2EI_{AB}}{L_{AB}} \left[ 2\theta_A - \theta_B \right] \left[ \frac{v_D - v_A}{L_{AB}} \right] + \text{FEM}_{AB}$$

(8)

This logic is incorrect and it is sometimes necessary to clarify that we do not loose the equation, we simply declare $M_{AB} = 0$ as if it were any other unknown. With the knowledge from above, EQN 8 can be written as:

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Once the substitutions are made for each equation as shown above, we have 8 equations with 8 unknowns. At this time, we have the opportunity to use one of several computer programs or hand held calculators to determine our internal end moments and joint rotations. For each individual member we also have two unknown end shears that we solve for these using statics. It is important to remember the sign convention used in the stiffness methods. The right hand rule/ counterclockwise is positive is a useful reminder. Positive moments are CCW as well as positive rotations. In the end we get the following results.

\[ M_{AB} = 0 \quad V_{AB} = 5.58 \text{ k} \]
\[ M_{BA} = -79.61 \quad V_{BA} = 9.42 \text{ k} \]
\[ M_{BC} = +79.61 \quad V_{BC} = 14.57 \text{ k} \]
\[ M_{CB} = -38.24 \quad V_{CB} = 35.43 \text{ k} \]
\[ M_{CD} = +38.24 \quad V_{CD} = 34.68 \text{ k} \]
\[ M_{DC} = -213.53 \quad V_{DC} = 27.82 \text{ k} \]

As well as \( \theta_A, \theta_B, \text{ and } \theta_C \).

The insight students get from this method stops once the equations of equilibrium are established. At this point, computers are used to solve the simultaneous equations. However, once the computer generates these internal end moments and joint rotations, the students are back to being engineers. Where do we apply these end moments? Which direction? How do we determine external reactions using these internal moments? Simple statics and joint equilibrium are used to establish our last two equations which allow us to calculate the external reactions at each support. The process of joint equilibrium will be used again in moment distribution as well as the direct stiffness. However, in the direct stiffness method, the external reactions are determined directly after solving for the unknown nodal deflections and rotations. Armed with nodal deflections and rotations, internal end moments and shears are easily solved for, but more on this later.

### III.B. Moment Distribution

Students rely on their engineering judgment to determine known and unknown DOF in the slope deflection method. For those students who want hands-on methods, physical models, as previously described in Figures 3 and 4, or graphical methods, moment distribution is more appealing and results in additional engineering intuition. Instead of talking degrees of freedom, we talk stiffness. If a moment is applied at the connection of two members, which members attract more moment (load)? We use several simple physical examples to demonstrate that the stiffer member attracts more force (Figure 6).
We start with a propped cantilever (Figure 2) to demonstrate this mathematically. Considering Figure 2, which is based on rotation at one end only, and slope deflection, one can see the distribution of moment as well as the carry over factor from one end of the member to the other (EQN 11 and 12).

\[
M_{zi} = \frac{2EI}{L} \left[ 2\theta_i + \theta_j - 3 \left( \frac{v_j - v_i}{L} \right) \right] + \text{FEM}_{ij} = \frac{4EI}{L} \theta_i \tag{11}
\]

\[
M_{zj} = \frac{2EI}{L} \left[ 2\theta_j + \theta_i - 3 \left( \frac{v_i - v_j}{L} \right) \right] + \text{FEM}_{ji} = \frac{2EI}{L} \theta_j \tag{12}
\]

If we have a continuous member over a support such as node B (Figure 5), both beam segments at the node will rotate the same distance, \( \theta \). The stiffness of a member is also represented by the \( 4EI/L \) quantity. Considering once again node B, the total stiffness of node B, \( k_B \), is the sum of the individual member stiffness, \( k_{BA} \) & \( k_{BC} \) at that node. The equation below demonstrates the total stiffness of node B (EQN 13).

At the node, moment is distributed according to each member’s relative stiffness. This relative stiffness is treated as a distribution factor (EQN 14). The higher relative stiffness, the higher the distribution of unbalanced moment will be to that member. This intuitively makes sense to most students. Moment will be distributed at that node by the relative stiffness of each member. We use the moment distributor (Figures 3 and 4) in class to demonstrate this phenomenon.

\[
k_{BA} = \frac{4EI_{AB}}{L_{AB}} \quad k_{BA} = \frac{4EI_{BA}}{L_{BA}} \quad k_B^{\text{TOTAL}} = \frac{4EI_{AB}}{L_{AB}} + \frac{4EI_{BA}}{L_{BA}} \tag{13}
\]

Relative stiffness (Distribution Factor) of a member is given by the equation:

\[
DF_{\text{member}} = \frac{k_{\text{member}}}{k_{jo \text{int}}} \tag{14}
\]

The basic propped cantilever diagram (Figure 2) also demonstrates the moment realized at the fixed end is half the moment at the roller support. This is true for those support conditions and prismatic beams only. Table 1 illustrates how students set up this...
graphical method. All loadings between the nodes are represented on the chart as fixed end moments. Support settlement is accounted for in the fixed end moments. The relatively high fixed end moments for segments BC and DC are due to the 0.25” support settlement of node C. Notice the higher distribution factor for BC relative to BA. This makes sense since the stiffness of member BC is greater due to the higher moment of inertia (Figure 5). When moments of inertia are equal, the increased length of CD makes the distribution factor for CB greater than that of CD. As stated earlier, an indirectly proportional relationship. This is not a paper on the moment distribution process, but we will note the migration of moment away from node A to node D. Roller at A has no rotational stiffness and thus retains zero moment. The rotationally stiff fixed support continues to attract moment throughout the process. Again, this is a very graphical process which helps some students see the logical migration of external load (internal moment) toward structural stiffness.

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<th>B</th>
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</tr>
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</table>

Table 1. Moment Distribution Example

The results make logical sense in that there is no moment at node A (end roller). There are equal and opposite moments on each face of the segments at nodes B and C. The largest moment is located at the fixed support (node D) in the stiffer portion of the member (segment BD).

IV. The Direct Stiffness Method

Once the students better understand structural behavior through structural analysis methods that can be solved by hand for small problems and compared to results from commercially available analysis programs, they are ready to look inside these commercially available analysis programs. Manual manipulation of the direct stiffness method using appropriate computer tools (e.g., MathCAD, Excel) allows the students to investigate mathematical modeling of connections, member types, etc.²

The direct stiffness method is extremely cool (students perspective) in that it allows students to solve directly for the external reactions, nodal displacements, and internal end moments and shears, where the slope deflection and moment distribution processes solve for only the internal end of member moments.

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Students learn how to create element stiffness matrices, how to assemble (stack) these matrices to form a structure force-displacement equation (how members are mathematically connected), and how to rearrange the structure force-displacement equation to simplify the solution for the unknown displacements and/or reactions.

The general steps using the direct stiffness problem are applied to the same example problem previously solved using the slope deflection and moment distribution methods. Students are encouraged to use software, such as Excel and MathCAD, to simplify the solution process.

- Global coordinate system is established and nodes and members are re-labeled with numbers and letters respectively. For example node A becomes node 1 and member AB is now member A (Figure 7).

Figure 7. Labeling of the Structure for the Direct Stiffness Method

- Force displacement equations are written for each element (Figure 8) using the generic format below (EQN 15). These equations relate end of member internal forces to the material properties, degrees of freedom, and the loading (fixed end effects) between the nodes. Loads between nodes create fixed-end effects and are accounted for in the \( \{\text{FEE}\} \) portion of EQN 15 (Member A, point load; Member B, Distributed load over half of member; Member C, ramp load; Figure 7 and 8).

Support settlement is accounted for by the \( \frac{v_j - v_i}{L} \) term used previously in both the slope deflection (EQN 3) and moment distribution methods (EQN 11).

\[
\begin{bmatrix}
F_{yi} \\
M_{zi} \\
F_{yi} \\
M_{zi}
\end{bmatrix} =
\begin{bmatrix}
\frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^2} & \frac{6EI}{L^2} \\
\frac{L^3}{6EI} & \frac{L^3}{4EI} & -\frac{L^3}{6EI} & \frac{L^3}{2EI} \\
-\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^2} & \frac{6EI}{L^2} \\
-\frac{L^3}{6EI} & \frac{L^3}{2EI} & -\frac{L^3}{6EI} & \frac{L^3}{4EI}
\end{bmatrix}
\begin{bmatrix}
V_i \\
\theta_i \\
V_j \\
\theta_j
\end{bmatrix} +
\begin{bmatrix}
\text{FEE}_{yi} \\
\text{FEM}_{zi} \\
\text{FEE}_{yi} \\
\text{FEM}_{zi}
\end{bmatrix}
\] (15)
MEMBER A ELEMENT FORCE-DISPLACEMENT RELATIONSHIP

| \(F_{y1}\) | 34.53 | 3729 | -34.53 | 3729 | \(v_1\) | 11.1 |
| \(M_{z1}\) | 3729 | 537037 | -3729 | 268519 | \(\theta_1\) | 480 |
| \(F_{y2}\) | -34.53 | -3729 | 34.53 | -3729 | \(v_2\) | 3.9 |
| \(M_{z2}\) | 3729 | 268519 | -3729 | 537037 | \(\theta_2\) | -240 |

MEMBER B ELEMENT FORCE-DISPLACEMENT RELATIONSHIP

| \(F_{y2}\) | 62.93 | 7552 | -62.93 | 7552 | \(v_2\) | 9.375 |
| \(M_{z2}\) | 7552 | 1208333 | -7552 | 604167 | \(\theta_2\) | 625 |
| \(F_{y3}\) | -62.93 | -7552 | 62.93 | -7552 | \(v_3\) | 40.625 |
| \(M_{z3}\) | 7552 | 604167 | -7552 | 1208333 | \(\theta_3\) | -1375 |

MEMBER C ELEMENT FORCE-DISPLACEMENT RELATIONSHIP

| \(F_{y3}\) | 32.22 | 4833 | -32.22 | 4833 | \(v_3\) | 43.75 |
| \(M_{z3}\) | 4833 | 966667 | -4833 | 483333 | \(\theta_3\) | 1875 |
| \(F_{y4}\) | -32.22 | -4833 | 32.22 | -4833 | \(v_4\) | 18.75 |
| \(M_{z4}\) | 4833 | 483333 | -4833 | 966667 | \(\theta_4\) | -1250 |

Figure 8. Element Force-Displacement Equations

- The element force-displacement equations are then stacked to create the structure force-displacement equation (Figure 9). Stacking combines the member stiffness of each element.

UNSORTED GLOBAL STRUCTURE FORCE-DISPLACEMENT RELATIONSHIP

| \(P_{y1}\) | 35 | 3729 | -35 | 3729 | 0 | 0 | 0 | 0 | \(v_1\) | 11.1 |
| \(P_{m1}\) | 3729 | 537037 | -3729 | 268519 | 0 | 0 | 0 | 0 | \(\theta_1\) | 480 |
| \(P_{y2}\) | -35 | -3729 | 97 | 3823 | -63 | 7552 | 0 | 0 | \(v_2\) | 13.28 |
| \(P_{m2}\) | 3729 | 268519 | 3823 | 1745370 | -7552 | 604167 | 0 | 0 | \(\theta_2\) | 385 |
| \(P_{y3}\) | 0 | 0 | -63 | -7552 | 95 | -2719 | -32.2 | 4833.3 | \(v_3\) | 84.38 |
| \(P_{m3}\) | 0 | 0 | 7552 | 604167 | -2719 | 2175000 | -4833.3 | 483333 | \(\theta_3\) | 500 |
| \(P_{y4}\) | 0 | 0 | 0 | 0 | -32 | -4833 | -32.2 | -4833.3 | \(v_4\) | 18.75 |
| \(P_{m4}\) | 0 | 0 | 0 | 0 | 4833 | 483333 | -4833.3 | 966667 | \(\theta_4\) | -1250 |

Figure 9. Structure Force-Displacement Equation

For this example (Figure 9), there are eight equations relating the eight unknowns. The external reactions (\(P_{y1}, P_{m1}\), etc.) are equal to the sum of the individual internal end of member forces, for example, \(P_{y2}\) can be expanded as shown below (EQN 16).

\[
P_{y2} = F_{y2}^A + F_{y2}^B = -34.53v_1 - 3729\theta_1 + (34.53 + 62.93)v_2 + (-3729 + 7552)\theta_2 - 62.93v_3 + 7552\theta_3 \tag{16}
\]
Unlike the previous methods, the direct stiffness method provides a direct solution for not only the internal moments, but the internal end of segment shear forces as well.

- The structure force displacement equation is then rearranged in the following manner. The known reactions (entire row, $P_{m1}$, $P_{m2}$, and $P_{m3}$ = 0 since there is no moment reaction at a roller) are moved up and then the known displacements (entire column, DOF) are moved to the right. Both of the movements are required to ensure the eight equations we started with remain the same — only the order of the external forces/moments and nodal deflection terms have changed in the series of equations. Notice that the matrix is still square, symmetric, and without negative terms on the diagonal (as was the case in Figure 9).

**SORTED AND PARTITIONED STRUCTURE FORCE-DISPLACEMENT EQUATION**

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<th>-3729</th>
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<td>0</td>
<td>$\theta_4=0$</td>
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</table>

Figure 10. Rearranged Structure Force-Displacement Equation

- The sorted equation is then partitioned as shown in Figure 10, which allows the force-displacement equations to be represented symbolically by the following two equations (EQN 17), where the only unknowns are $\{\delta_f\}$ and $\{P_s\}$:

$$\begin{align*}
\begin{bmatrix}
P_f \\
P_s
\end{bmatrix} &=
\begin{bmatrix}
K_{ff} & K_{fs} \\
K_{sf} & K_{ss}
\end{bmatrix}
\begin{bmatrix}
\delta_f \\
\delta_s
\end{bmatrix} +
\begin{bmatrix}
P_{f, \text{fixed}} \\
P_{s, \text{fixed}}
\end{bmatrix}
\end{align*}$$

(17)

The unknown displacements, $\{\delta_f\}$, can be solved directly by as shown (EQN 18):

$$\{\delta_f\} = \left(K_{ff}\right)^{-1}\left(\{P_f\} - \{P_{f,\text{fixed}}\} - \left[K_{fs}\right]\{\delta_s\}\right)$$

(18)

The unknown reactions, $\{P_s\}$, can then be solved directly using the following relationship, once $\{\delta_f\}$ is known:

$$\{P_s\} = \left[K_{sf}\right]\{\delta_f\} + \left[K_{ss}\right]\{\delta_s\} + \{P_{s,\text{fixed}}\}$$

(19)
MathCAD is used to solve for the unknown displacements and the unknown reactions. The solution is shown below:

\[
\begin{align*}
K_{ff} &:= \begin{pmatrix} 537037 & 268519 & 0 \\ 268519 & 1745370 & 604167 \\ 0 & 604167 & 2175000 \end{pmatrix}, & \quad K_{fs} &:= \begin{pmatrix} 3729 & -3729 & 0 & 0 & 0 \\ 3729 & 3823 & -7552 & 0 & 0 \\ 0 & 7552 & -2719 & 483333333 & 483333333 \\
\end{pmatrix},
\end{align*}
\]

\[
K_{sf} := \begin{pmatrix} 3729 & 3729 & 0 \\ 3729 & 3823 & 7552 \\ 0 & -7552 & -2719 \\ 0 & 0 & -4833 \\ 0 & 0 & 4833333 \\
\end{pmatrix}, & \quad K_{ss} := \begin{pmatrix} 35 & -35 & 0 & 0 & 0 \\ -35 & 97 & -63 & 0 & 0 \\ 0 & -63 & 95 & -3222222222 & 483333333 \\ 0 & 0 & -32 & 3222222222 & -483333333 \\ 0 & 0 & 4833 & -483333333 & 9666666667 \\
\end{pmatrix},
\]

\[
P_f := \begin{pmatrix} 0 \\ 0 \\ 0 \\
\end{pmatrix}, \quad \Delta_s := \begin{pmatrix} 0 \\ -0.25 \\ 0 \\ 0 \\
\end{pmatrix}, \quad P_{f\text{fixed}} := \begin{pmatrix} 480 \\ 385 \\ 500 \\
\end{pmatrix}, \quad P_{sf\text{fixed}} := \begin{pmatrix} 11.1 \\ 13.275 \\ 84.375 \\ 18.75 \\ -1250 \\
\end{pmatrix},
\]

\[
\Delta_f := K_{ff}^{-1} \left( P_f - K_{fs} \Delta_s - P_{f\text{fixed}} \right), \quad \Delta_f = \begin{pmatrix} -3.03 \times 10^{-4} \\ -1.182 \times 10^{-3} \\ -2.142 \times 10^{-4} \end{pmatrix},
\]

\[
P_s := K_{sf} \Delta_f + K_{ss} \Delta_s + P_{sf\text{fixed}}, \quad P_s = \begin{pmatrix} 5.564 \\ 24.02 \\ 70.13 \\ 27.785 \\ -2.562 \times 10^3 \end{pmatrix},
\]

\[
P_y1 = 5.56 \text{K (up)}, \quad P_y2 = 24.00 \text{K (up)}, \quad P_y3 = 70.13 \text{K (up)}, \quad P_y4 = 27.78 \text{K (up)}, \quad P_{m4} = 213 \text{K - f(CW)}
\]

Figure 11. Partial MathCAD Solution
VI. Assessment

Does it work? We teach it so it must be valuable. Our students do very well on the Fundamentals of Engineering Exam and in graduate school, so it must work. That might be enough, but we did not stop there. We use web-based assessment data provided through the institution-wide end of the semester survey system to answer this question. The survey not only assesses how well the instructor does their job, but how well the course is organized. We include the list of course objectives to determine how well the course accomplished its stated mission in preparing the students to accomplish certain tasks. The students respond to the list of course objectives (Figures 12 and 13) with 1 being strongly disagree and 5 being strongly agree.

All of our Civil Engineering majors take our first Structural Analysis Course, CE403. They see all of the classical methods before covering the direct stiffness method in the last 4 lessons of the course. The students feel that they understand it as well as any of the methods presented (Question D19, Figure 12). Reason – we focus on pointing out the structural behavior that can be learned from the classical hand methods. In the Advanced Structural Analysis course (CE491), which is an elective that most of our majors take, all of the classical methods are reviewed at blinding speed with an associated homework. At each step the important points concerning structural behavior are refreshed. Upon looking at the associated assessment data for the course objectives (Question D6, Figure 13), we see that the most understood concept is the Direct Stiffness method. If the classical methods were a waste of time in understanding structural behavior, then more students would agree strongly that we review too much material from CE403 in CE491 (Question D18, Figure 13). Therefore, the assessment data shows in a non-direct way that the classical methods are needed to provide a fuller understanding of the behavior of structures.

Teaching the classical methods allows us to more effectively appeal to the different learning styles students have. The repetition of using the classical methods before introducing the direct stiffness method helps some students learn more effectively. Some students are able to understand the direct stiffness method better because they learn the underlining concepts during their lessons on the classical methods (serves as bite-sized chunks). Using the classical methods also enables students to see how a structure can be analyzed using a series of equations (slope deflection method) and a graphical method (moment distribution). Hand calculations are used throughout each of the classical methods to show how the processes are related and to show what is physically happening with the structure.
Figure 12. CE403 Course Assessment Feedback

Figure 13. CE491 Course Assessment Feedback
VI. Conclusion

Our students greatly appreciate the fact that the direct stiffness method allows them to solve directly for the external reactions and/or displacements. Slope deflection and moment distribution methods on the other hand only solve for internal end of member moments. The students then have to solve for internal end of member shear forces by hand. Finally, nodal equilibrium is applied to solve for the external reactions. The time required to solve even simple problems is a huge issue, especially when not using the direct stiffness method and associated computer support. It is still essential to teach the classical methods as they are used to develop the direct stiffness method.

The direct stiffness method naturally ties the concepts of how member properties, degrees of freedom, and loading all affect internal member shear forces and moments, and ultimately external reactions, but maybe with too much information all at once if we started with the direct stiffness first. By teaching students the classical methods first, the necessary connectivity and knowledge base for transition to the direct stiffness method is possible.

Multiple methods allow for repetition, varying learning styles, and gradual insight into the world of structural analysis. Understanding and reflecting on the assumptions inherent in the direct stiffness method is possible through the gradual traverse through the classical methods, rather than just using commercial packages. Additionally, a historical perspective and deeper appreciation of structural analysis is only possible through exploration of the Classical Structural Analysis Methods, not to mention understanding and appreciating the computer tools that are now available. Starting with a black-box approach would be boring, oversimplifies the effort, and generally limits full understanding of structural behavior and what exactly affects structural behavior.
Bibliography

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